Review of Random Variables and Distributions

- **Random variables**
- **Measure of central tendencies and variability (means and variances)**
- **Joint density functions and independence**
- **Measures of association (covariance and correlation)**
- **Interesting result**
- **Conditional distributions**
- **The Normal distribution**
- **Appendix I: Correlation, independence and linear relationships**
- **Appendix II: Covariance and independence**

1. Random variables

a. Random variables: X, Y, Z … take on different values with different probabilities; convention is to use capital letters for random variables and lower case letters for realized values

   i. So, for instance, X is a random variable, and x or $x_1, x_2,$ *and* x_3 would be specific realized values of X

b. (Probability) Density functions (pdfs): describe the distribution of the random variable … the probability that the random variable takes on different values… used to determine probabilities

   i. Discrete random variable (e.g. Binomial distribution): takes on a finite or countably infinite set of values with positive probability

      1. density function: $f(x_j) = P(X = x_j) \geq 0 \, and \sum f(x_j) = 1$ (note *sigma* notation)

   ii. Continuous random variable (e.g. Normal distribution)

      1. density function: $f(x) \geq 0 \quad and \quad \int f(x)dx = 1$

   iii. Use the density functions to determine the probabilities:

      1. Discrete: $P(a < X \leq b) = \sum_{a<x\leq b} P(X = x) = \sum_{a<x\leq b} f(x)$

      2. Continuous: $P(a < X \leq b) = \int_a^b f(x)dx$

c. Examples of random variables

i. Uniform [a,b]: $f(x) = \frac{1}{b-a}$ $x \in [a,b]$ and is 0 otherwise

ii. Standard Normal - N(0,1): $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

2. Measures of central tendencies and variability

a. Expectation/Mean (measure of central tendency): $E(X)$, μ

i. The average value of X (observed with a large number of random samples from the distribution)

ii. A weighted average of the different values of X (weight the values by their respective probabilities)

1. Discrete: $E(X) = \mu = \sum x_i P(X = x_i) = \sum x_i f(x_i)$

2. Continuous: $E(X) = \mu = \int xf(x)dx$

iii. Properties

1. Linear operator: $E(aX + b) = aE(X) + b$

a. Extends to many random variables:

$E(\sum a_i X_i) = \sum E(a_i X_i) = \sum a_i E(X_i) = \sum a_i \mu_i$

2. And for some function g(.), $E(g(X)) = \sum g(x_i) f(x_i)$ or $\int g(x) f(x) dx$ for a continuous distribution

b. Variance (measure of variability or dispersion around the mean): $Var(X)$, σ^2

i. The average squared deviation of X from its mean (observed with a large number of random samples from the distribution)

ii. A weighted average of the different squared deviations of X from its mean (weight the squared deviations by their respective probabilities)

1. Discrete: $Var(X) = E(X-\mu)^2 = \sum \left(x_i - \mu\right)^2 P(X = x_i) = \sum \left(x_i - \mu\right)^2 f(x_i)$

2. Continuous: $Var(X) = E(X-\mu)^2 = \int \left(x - \mu\right)^2 f(x)dx$

3. $\sigma^2 = E(X-\mu)^2 = E(X^2) - \mu^2$

iii. Properties:

1. Not a linear operator: $Var(aX + b) = a^2 Var(X)$

iv. Standard deviation (StdDev): $\sigma = \sqrt{\sigma^2}$ … (positive square root)

1. Linear operator: if a>0, then

$$StdDev(aX+b)=\sqrt{Var(aX)}=\sqrt{a^2Var(X)}=a\ StdDev(X)$$

c. Standardizing random variables (z-scores): $Z=\dfrac{X-\mu}{\sigma}$ (has mean zero and unit variance)

i. Mean: $E(Z)=E\left(\dfrac{X-\mu}{\sigma}\right)=\dfrac{1}{\sigma}\left(E(X)-\mu\right)=0$

ii. Variance: $Var(Z)=E(Z^2)=\dfrac{1}{\sigma^2}Var(X)=1$

3. Joint density functions

a. Consider X and Y, two random variables (e.g. people are randomly drawn from a population and their heights and weights are recorded)

b. If discrete, then the joint density is defined by $f_{XY}(x,y)=P(X=x\ \&\ Y=y)$

c. Note that $P(X=x)=f_X(x)=\sum_y P(X=x\ \&\ Y=y)=\sum_y f_{XY}(x,y)$.

i. So, the marginal density $P(X=x)=f_X(x)$ is the sum over the joint densities $\sum_y f_{XY}(x,y)$.

d. Here's an example.

i. In the following table, the random variable X takes on three values (x1, x2 and x3), and Y takes on two (y1 and y2). The figures in the XY box are the joint probabilities, $f_{XY}(x,y)=P(X=x\ \&\ Y=y)$. And so, for example, $f_{XY}(x1,y1)=P(X=x1\ \&\ Y=y1)=.2$.

ii. And the marginal probabilities can be recovered from the joint probabilities by just summing across the rows and columns. So, for example, $P(X=x1)=f_X(x1)=\sum_{j=1,2}P(X=x1\ \&\ Y=yj)$
$=f_{XY}(x1,y1)+f_{XY}(x1,y2)=.2+.2=.4$.

		Y: y1	Y: y2
X	x1	0.2	0.2
	x2	0.1	0.3
	x3	0.1	0.1

0.4	P(X=x1)
0.4	P(X=x2)
0.2	P(X=x3)

0.4	*0.6*
P(Y=y1)	P(Y=y2)

e. ***Independence***

i. $f_{X,Y}(x,y) = P(X = x)P(Y = y) = f_X(x) f_Y(y)$ for all values of X and Y, (x,y) … the joint density function is the product of the *marginal* densities (applies to discrete and continuous distributions)

1. X and Y in the previous example are not independent, since, for example:

$$f_{X,Y}(x1, y1) = .2 \neq P(X = x1)P(Y = y1) = f_X(x1) f_Y(y1) = (.4)(.4) = .16$$

ii. We can extend to many independent random variables:

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$
$$= f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_n}(x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

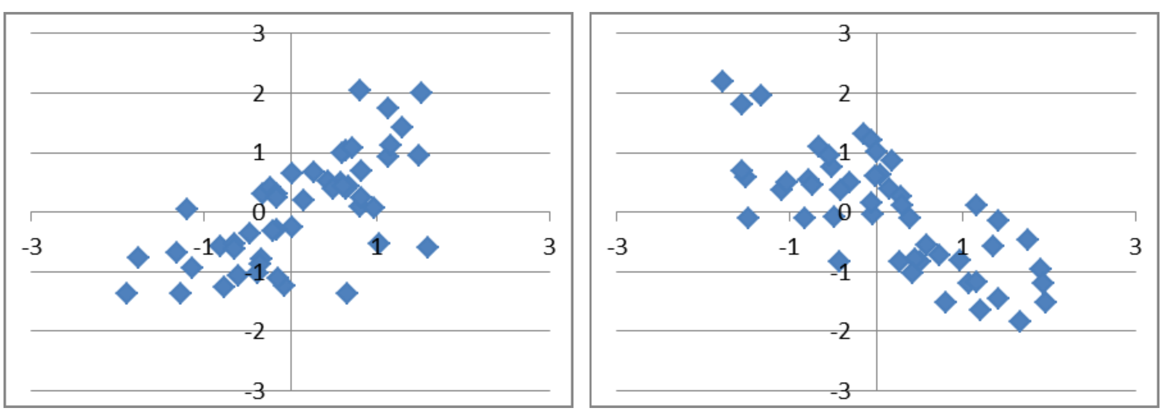
iii. Not independent means *dependent*

4. Measures of association

a. Consider two random variables, X and Y.

b. Covariance: $Cov(X,Y) = \sigma_{XY} = E(X - \mu_X)(Y - \mu_Y) = \sum (x - \mu_X)(y - \mu_Y) f(x,y)$

c. Some examples: X and Y both have mean 0 in the following examples. On the left, most of the data are in quadrants I and III, where $(x - \mu_X)(y - \mu_Y) > 0$, and so when you sum those products you get a positive covariance. Most of the action on the right is in quadrants II and IV where $(x - \mu_X)(y - \mu_Y) < 0$, and so those products sum to a negative covariance.



d. Properties:

i. $Cov(X,Y) = \sigma_{XY} = E(XY) - \mu_X \mu_Y$

ii. Note that $Cov(X,X) = \sigma_{XX} = E(X - \mu_X)(X - \mu_X) = Var(X) = \sigma_X^2$

iii. Measures the extent to which there is a **linear** relationship between X and Y

iv. If $Cov(X,Y) > 0$ then as illustrated above, X and Y tend to move together in a positive direction, so that increases is X are on average associated with increases in Y… and if the covariance is negative, then they tend to move in opposite directions

v. If X and Y are independent, then $Cov(X,Y) = \sigma_{XY} = 0$

1. Opposite need not hold… $\sigma_{XY} = 0$ does not necessarily imply independence… it could just mean that there is a highly non-linear relationship between X and Y.
2. Here's an example of X & Y having zero covariance, but not being independent:

Joint & Marginal Densities

		Y = 0	Y = 1		
X	-1	-	0.33	*0.33*	E(X)=
	0	0.33	-	*0.33*	0
	1	-	0.33	*0.33*	
		0.33	*0.67*		
	E(Y)=		0.67		

Cov Contributions

	Y = 0	Y = 1
-1	0.67	(0.33)
0	-	-
1	(0.67)	0.33

Cov(X,Y) 0.0000

vi. $Cov(a+bX, c+dY) = bdCov(X,Y)$

vii. $|\sigma_{XY}| \le |\sigma_X \sigma_Y|$ the magnitude of the covariance is never greater than the product of the magnitudes of the standard deviations (this is an instance of the Cauchy-Schwartz Inequality)

e. Variances of sums of random variables

i. $Var(X+Y) = \sigma_X^2 + 2Cov(X,Y) + \sigma_Y^2$

ii. More generally: $Var(a_1X_1 + a_2X_2) = a_1^2\sigma_{X_1}^2 + 2a_1a_2Cov(X_1, X_2) + a_2^2\sigma_{X_2}^2$

iii. So if $Cov(X,Y) = 0$ (so that X and Y are *uncorrelated*), then $Var(X+Y) = Var(X) + Var(Y)$ (the variance of the sum is the sum of the variances)

iv. And even more generally:

1. $Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov(X_i, X_j)$ … note that when i=j, the term is $a_i^2 Cov(X_i, X_i) = a_i^2 \sigma_i^2$
2. If the X_i's are pairwise uncorrelated, then $Cov(X_i, X_j) = 0$ when $i \ne j$, and so in this case, $Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov(X_i, X_j) = \sum_{i=1}^{n} a_i a_i Cov(X_i, X_i) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

a. If they are pairwise uncorrelated, then the variance of the sum is the sum of the variances.

f. Correlation: $Corr(X,Y) = \rho_{XY} = \dfrac{Cov(X,Y)}{StdDev(X)\,StdDev(Y)} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

i. $\left|\sigma_{XY}\right| \le \left|\sigma_X \sigma_Y\right| \Rightarrow -1 \le \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y} \le 1$ … so $-1 \le \rho_{XY} \le 1$

ii. And similar to above:

1. If $Cov(X,Y) = 0$, then $\rho_{XY} = 0$.
2. If X and Y are independent, then they are uncorrelated and $\rho_{XY} = 0$
3. ρ_{XY} captures the extent to which there is a **<u>linear</u>** relationship between X and Y … which is similar to, though not the same as, the extent to which they move together
4. If $Y = aX + b$, then $Corr(X,Y) = \rho_{XY} = \dfrac{Cov(X,Y)}{StdDev(Y)\ StdDev(X)}$

$$= \frac{aVar(X)}{|a|\,\sigma_X \sigma_X} = \frac{a}{|a|} = 1 \; or \; -1 \; \ldots$$

and so if X and Y are linearly related they have a correlation of +1 or -1.

iii. Properties:

1. $Corr(a_1 X_1 + b_1, a_2 X_2 + b_2) = Corr(X_1, X_2)$ if $a_1 a_2 > 0$, and $= -Corr(X_1, X_2)$ if $a_1 a_2 < 0$
2. So linear transformations of random variables may affect the sign of the correlation, but not the magnitude.

5. Interesting result

a. Suppose that the random variable Y is a linear function of another random variable X plus an additive random error U, which is uncorrelated with X, then:

i. $Y = a + bX + U$, where $Y, X \; and \; U$ are all random variables and $Cov(X,U) = 0$

ii. $Cov(X,Y) = Cov(X, a + bX + U) = Cov(X,a) + bCov(X,X) + Cov(X,U)$

iii. Since $Cov(X,a) = Cov(X,U) = 0$, $Cov(X,Y) = bCov(X,X)$

iv. … or $b = \dfrac{Cov(X,Y)}{Cov(X,X)} = \dfrac{\sigma_{XY}}{\sigma_{XX}} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}\dfrac{\sigma_Y}{\sigma_X} = \rho_{XY}\dfrac{\sigma_Y}{\sigma_X} = Corr(X,Y)\dfrac{StdDev(Y)}{StdDev(X)}$

1. This is a relationship that will haunt you throughout the semester.

6. Conditional distributions

a. Recall the definition of conditional probabilities: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, which might suggest that $P(Y = y \mid X = x) = \dfrac{P(Y = y \;\&\; X = x)}{P(X = x)}$

b. If discrete, then $f_{Y|X}(y \mid x) = P(Y = y \mid X = x) = \dfrac{f_{X,Y}(x, y)}{f_X(x)}$ … same formula applies to continuous distributions

   i. Dividing by $f_X(x)$ effectively "scales up" the marginal densities…. and ensures that you have a valid density function, since

$$\int f_{Y|X}(y \mid x)dy = \int \frac{f_{X,Y}(x, y)}{f_X(x)} dy = \frac{1}{f_X(x)} \int f_{X,Y}(x, y)dy = \frac{f_X(x)}{f_X(x)} = 1.$$

c. If X and Y are independent then the conditional distributions and marginal distributions are the same

   i. $f_{Y|X}(y \mid x) = f_Y(y)\; and = f_{X|Y}(x \mid y) = f_X(x)$

   ii. In words: If X and Y are independent than knowing the particular value of Y, y, tells you nothing new about X, and vice-versa

d. Conditional expectations and variances

   i. The expected value of Y conditional on X being a certain value… as the value of X changes, the conditional expectation of Y given X=x may also change

      1. $E(Y \mid X = x) = E(Y \mid x) = \sum y_j P(Y = y_j \mid X = x) = \sum y_j f_{Y|X}(y_j \mid x)$

      2. If X and Y are independent, then $E(Y \mid X = x) = E(Y)$ … knowing the value of X doesn't change the expected value of Y

   ii. Conditional variances are similarly defined… the expected squared deviation from the conditional mean:

      1. $Var(Y \mid X = x) = E(\left[Y - E(Y \mid X = x)\right]^2 \mid X = x) = E(Y^2 \mid x) - \left(E(Y \mid x)\right)^2$

7. The Normal distribution

a. Standard Normal (Gaussian): $\mathrm{N}\left(\mu,\sigma^2\right)$ has mean μ and variance σ^2

b. If X is $\mathrm{N}\left(\mu,\sigma^2\right)$, then $Z = \dfrac{X-\mu}{\sigma}$ is $\mathrm{N}\left(0,1\right)$ (the Standard Normal distribution)

c. Properties:

i. If X is $\mathrm{N}\left(\mu,\sigma^2\right)$ then $aX + b \sim \mathrm{N}\left(a\mu + b, a^2\sigma^2\right)$

ii. If X_1 and X_2 are independent with the same distribution, $\mathrm{N}\left(\mu,\sigma^2\right)$, then $X_1 + X_2 \sim \mathrm{N}(2\mu, 2\sigma^2)$

1. This implies that $\frac{1}{2}\left(X_1 + X_2\right) \sim \mathrm{N}(\mu, \frac{1}{2}\sigma^2)$.

iii. More generally, assume that n random variables ($X_1, X_2, \ldots X_n$) are independently and identically distributed $\mathrm{N}\left(\mu,\sigma^2\right)$, then $\sum X_i \sim \mathrm{N}(n\mu, n\sigma^2)$ and $\bar{X} = \frac{1}{n}\sum X_i \sim \mathrm{N}(\mu, \frac{1}{n}\sigma^2)$.

iv. $\bar{X} = \frac{1}{n}\sum X_i$ is a specific form of the more general weighted average $Y = \sum \alpha_i X_i$, where $0 \le \alpha_i \le 1$ for all i and $\sum \alpha_i = 1$.

1. Y will have mean $\sum \alpha_i \mu = \mu \sum \alpha_i = \mu$
2. … and variance $= \sum \alpha_i^2 \sigma^2 = \sigma^2 \sum \alpha_i^2$, and will be Normally distributed.

8. **Appendix I - Correlation and Linear Relationships:**

   $|\rho_{XY}| = 1 \Leftrightarrow P(Y = \beta_0 + \beta_1 X) = 1$

   a. Linear implies a correlation of +1 or -1

      i. Suppose that $Y = \beta_0 + \beta_1 X$ and $\beta_1 \neq 0$.

      ii. Then $\text{cov}(X,Y) = \text{cov}(X, \beta_0 + \beta_1 X) = E((X - \mu_X)(\beta_0 + \beta_1 X - \beta_0 - \beta_1 \mu_X))$
      $= \beta_1 E((X - \mu_X)^2) = \beta_1 \text{ var}(X)$.

      iii. And since $\text{var}(Y) = E((\beta_0 + \beta_1 X - \beta_0 - \beta_1 \mu_X)^2) = \beta_1^2 E((X - \mu_X)^2) = \beta_1^2 \text{ var}(X)$, the correlation of X and Y is:
      $\rho_{XY} = \dfrac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \dfrac{\beta_1 \text{ var}(X)}{\sqrt{\text{var}(X)\beta_1^2 \text{ var}(X)}} = \dfrac{\beta_1}{\sqrt{\beta_1^2}} = +1 \; or \; -1$ depending on the sign of $\beta_1 \neq 0$.

   b. Non-linear implies correlation not +1 or -1 … here's an example:

      i. Suppose that $U = Y - (\beta_0 + \beta_1 X)$, where $\mu_U = 0 \; and \; \text{cov}(X,U) = 0$, but $\text{var}(U) = \sigma_U^2 \neq 0$ (so we don't have a perfectly linear relationship between *X* and *Y*).

      ii. Then $\text{cov}(X,Y) = \text{cov}(X, \beta_0 + \beta_1 X + U) = E((X - \mu_X)(\beta_0 + \beta_1 X + U - \beta_0 - \beta_1 \mu_X))$.

      iii. And since $\text{var}(Y) = E((\beta_0 + \beta_1 X + U - \beta_0 - \beta_1 \mu_X)^2)$
      $= \beta_1^2 E((X - \mu_X)^2) + 2\beta_1 \text{ cov}(X,U) + \text{var}(U) \; = \beta_1^2 \text{ var}(X) + \sigma_U^2$, the correlation of *X* and *Y* is: $\rho_{XY} = \dfrac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \dfrac{\beta_1 \text{ var}(X)}{\sqrt{\text{var}(X)\left(\beta_1^2 \text{ var}(X) + \sigma_U^2\right)}}$.

      iv. Since $\text{var}(U) = \sigma_U^2 \neq 0$, the denominator will be larger in magnitude than the numerator and so $|\rho_{XY}| < 1$.

      v. Notice that if $\sigma_U^2 = 0$, then we have a linear relationship, and as above $\rho_{XY} = +1 \; or \; -1$.

9. Appendix II: Covariance and independence

Not Independent!

		Y = X^2			marginal for X
		0	**0.25**	**1**	
X	**-1**	0%	0%	20%	20%
	-0.5	0%	20%	0%	20%
	0	20%	0%	0%	20%
	0.5	0%	20%	0%	20%
	1	0%	0%	20%	20%

marginal for Y

20%	40%	40%

Independent!

		Y			marg
		0	**0.25**	**1**	
X	**-1**	4%	8%	8%	20%
	-0.5	4%	8%	8%	20%
	0	4%	8%	8%	20%
	0.5	4%	8%	8%	20%
	1	4%	8%	8%	20%
	marg	20%	40%	40%	**Indep!**

Covariance calculation

prob	X	Y
20%	-1	1
20%	-0.5	0.25
20%	0	0
20%	0.5	0.25
20%	1	1

	X	Y
mean	**0**	**0.5**
variance	**0.625**	**0.2188**
covariance		0

covar = 0

X-muX	Y-muY		product
-1	0.5		-0.5
-0.5	-0.25		0.125
0	-0.5		0
0.5	-0.25		-0.125
1	0.5		0.5